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A contribution to the theory and practice of multiple time scales expansion of nonlinear oscillators

Yitzhak Weissman

Soreq Nuclear Research Centre, Yavne, Israel and Department of Chemistry, Tel Aviv University, Tel Aviv, Israel

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Abstract. In this paper we present a systematic method for the generation of multiple time scales expansion of the oscillator

$$(\mathrm{d}^2x/\mathrm{d}t^2)+\omega^2x=\epsilon\sum_{n=1}^{N}\sum_{m=1}^{m}g_{nm}x^n(\mathrm{d}x/\mathrm{d}t)^m$$

to any order. The excessive freedom which is inherent in the process is conveniently controlled, thus allowing one to generate easily different expansions to the same problem. This option was used to study the extent by which different uses of this freedom can affect the accuracy of the expansion, concluding that the effect may be significant. The new method was applied to the Duffing and the van der Pol oscillators. The complicated algebraic computations involved were accomplished by a computer.

1. Introduction

A straightforward application of the expansion

$$x=x_0+\epsilon x_1+\epsilon^2x_2+\ldots+\epsilon^Nx_N+R_N \tag{1.1}$$

to the perturbed harmonic oscillator

$$\mathrm{d}^2x/\mathrm{d}t^2+\omega^2x=\epsilon g(x,\mathrm{d}x/\mathrm{d}t) \tag{1.2}$$

yields a remainder R_N which is of the order of $(\epsilon t)^{N+1}$, a fact which renders this procedure useless for times longer than ϵ^{-1} regardless of N. This hindrance stems from the appearance of secular terms (terms which are proportional to positive powers of t) in the x_i. In order to eliminate the secular terms, one has to introduce extra freedom in the expansion (1.1). This can be done by introducing auxiliary variables; expansions using this technique are called multivariable expansions (Nayfeh 1973) and their Nth-order remainder is estimated by $\epsilon^{N+1}t$.

We shall consider here one version of multivariable expansions, namely the multiple time scales expansion (MTSE), which is based on the introduction of auxiliary time variables scaled by powers of ϵ:

$$T_0=t,\qquad T_1=\epsilon t\ldots,\qquad T_N=\epsilon^Nt \tag{1.3}$$

(Frieman 1963, Nayfeh 1965, 1968, Sandri 1965, 1967). This method has been applied successfully to various physical problems, recently including problems in atomic and molecular physics (Wong *et al* 1976). The application of multivariable expansions in general and the MTSE in particular to the perturbed harmonic oscillator problem has been studied already in considerable detail (Nayfeh 1965, 1973, Reiss 1971, Levine and Lubot 1975), but until recently no systematic procedure for that application has been proposed. This may be due to the fact that multivariable expansions beyond second order were seldom attempted, as the amount of algebraic computation becomes prohibitively large in higher orders. Nowadays, using a language capable of performing symbolic algebraic computations, this tedious task can be accomplished with the help of a computer. However, in order to use a computer to generate an MTSE, one has to devise a completely systematic procedure for the application of this method. The MTSEs presented in this article for the Duffing and the van der Pol oscillators were generated by a computer program based on the procedure described in the following section.

Recently, progress in the systematisation of different multivariable expansions was reported. Eminhizer *et al* (1976) devised a systematic method to treat a system of coupled anharmonic oscillators which has remarkable convergence properties and avoids both secular terms and small denominators. Melvin (1977) suggested a different method to treat a single anharmonic oscillator. Both methods are limited to conservative systems, and their application requires arithmetical calculation only; therefore those methods can be coded using a conventional computer language like FORTRAN.

The freedom introduced by the auxiliary variables is in general more than enough to eliminate the secular terms; the extra freedom is usually discarded by certain arbitrary decisions. Different decisions in handling the extra freedom may lead to different expansions with different degrees of accuracy. This subject is discussed in § 4; however, the question of how to use the extra freedom for the benefit of the expansion is still open.

2. A systematic prescription for the application of the MTSE method

We start by introducing a complex variable z:

$$z = \exp(-\mathrm{i}\omega t)(\mathrm{d}x/\mathrm{d}t + \mathrm{i}\omega x) \tag{2.1}$$

(this convenient transformation has been used also by Montroll and Helleman (1976)). It is easily verified that

$$\exp(\mathrm{i}\omega t)\,\mathrm{d}z/\mathrm{d}t = \mathrm{d}^2x/\mathrm{d}t^2 + \omega^2 x \tag{2.2}$$

$$x = (2\mathrm{i}\omega)^{-1}[\exp(\mathrm{i}\omega t)z - \exp(-\mathrm{i}\omega t)\bar{z}] \tag{2.3}$$

$$\mathrm{d}x/\mathrm{d}t = \tfrac{1}{2}(\exp(\mathrm{i}\omega t)z + \exp(-\mathrm{i}\omega t)\bar{z}). \tag{2.4}$$

Though our method can be formulated for a general nonlinearity $g(x, \mathrm{d}x/\mathrm{d}t)$, we shall limit ourselves to the case in which g is a polynomial both in x and $\mathrm{d}x/\mathrm{d}t$, i.e.

$$g(x, \mathrm{d}x/\mathrm{d}t) = \sum_{n=0}^{N} \sum_{m=0}^{M} g_{nm} x^n (\mathrm{d}x/\mathrm{d}t)^m. \tag{2.5}$$

With this limitation several simplifications occur, and anyway this is the form of g usually encountered in practice. Substituting the right-hand sides of equations (2.3) and (2.4) for x and $\mathrm{d}x/\mathrm{d}t$ in g respectively, we obtain a new function $h(z, \bar{z}, \exp(\mathrm{i}\omega t))$. In terms of the new variables, equation (1.2) takes the form

$$\mathrm{d}z/\mathrm{d}t = \epsilon \exp(-\mathrm{i}\omega t)h(z, \bar{z}, \exp(\mathrm{i}\omega t)). \tag{2.6}$$

At this point we introduce the time scales (1.3) and the Nth-order expansion

$$z=z_0(T_0\ldots T_N)+\sum_{j=1}^{N}\epsilon^j z_j(z_0,\bar{z}_0,\exp(\mathrm{i}\omega T_0)). \tag{2.7}$$

Substituting equation (2.7) into equation (2.5) and treating the T_j as independent variables, we get

$$\sum_{j,k=0}^{N}\epsilon^{j+k}\,\partial z_j/\partial T_k=\epsilon\exp(-\mathrm{i}\omega T_0)h(z,\bar{z},\exp(\mathrm{i}\omega T_0)). \tag{2.8}$$

The right-hand side of equation (2.8) can be expanded as a power series in ϵ; let us denote the coefficient of ϵ^m in it by F_m. It is easily verified that

$$F_0=0 \tag{2.9}$$

$$F_m=F_m(z_0,\bar{z}_0,\ldots,z_{m-1},\bar{z}_{m-1},\exp(\mathrm{i}\omega T_0)). \tag{2.10}$$

Equating the coefficients of equal powers of ϵ on both sides of equation (2.8) we obtain

$$\sum_{j=0}^{m}\partial z_j/\partial T_{m-j}=F_m \tag{2.11}$$

for $m=0,1\ldots N$. The $m=0$ equation in this system states that $\partial z_0/\partial T_0=0$, and therefore T_0 can be omitted from the list of the arguments of z_0 in equation (2.7). This system of equations, together with the initial conditions, is used to determine the expansion coefficients $z_0\ldots z_N$. It is completely equivalent to the system of equations which appear in MTSEs in earlier works (Nayfeh 1973), but is formally simpler since the second derivatives have been eliminated. This important virtue of the system (2.11) simplifies significantly the subsequent treatment.

Before presenting the procedure for the systematic solution of the system (2.11) we make some preliminary remarks and introduce some concepts and notations. First, we draw attention to the fact that z_0 plays in the expansion a completely different role from the role of the functions $z_1\ldots z_N$. This fact is reflected already in the arguments of the functions: the arguments of z_0 are $T_1\ldots T_N$ while the arguments of $z_1\ldots z_N$ are z_0, $\bar{z}_0$ and $\exp(\mathrm{i}\omega T_0)$. We shall consider the function z_j, $j=1\ldots N$ as 'determined' if its dependence on z_0, $\bar{z}_0$ and $\exp(\mathrm{i}\omega T_0)$ is established; in particular, it will be considered as such even if the dependence of z_0 on part or all the time scales is still unknown.

A function of $f(T_0)$ will be called a generalised power series in $\exp(\mathrm{i}\omega T_0)$ if it can be represented as:

$$f(T_0)=\sum_{k=-m}^{n}a_k\exp(\mathrm{i}k\omega T_0). \tag{2.12}$$

It is easily verified that F_m is a generalised power series in $\exp(\mathrm{i}\omega T_0)$. We define the '$T_0$ average', $\langle f\rangle$, of f as

$$\langle f\rangle=a_0. \tag{2.13}$$

Finally, we introduce the function $G_m(z_0,\bar{z}_0,\exp(\mathrm{i}\omega T_0))$:

$$\begin{aligned}(G_m(z,\bar{z}_0,&\exp(\mathrm{i}\omega T_0))\\ &\equiv F_m(z_0,\bar{z}_0,z_1(z_0,\bar{z}_0,\exp(\mathrm{i}\omega T_0)),\\ &\quad\bar{z}_1(z_0,\bar{z}_0,\exp(\mathrm{i}\omega T_0))\ldots z_{m-1}(z_0,\bar{z}_0,\exp(\mathrm{i}\omega T_0),\\ &\quad\bar{z}_{m-1}(z_0,\bar{z}_0,\exp(\mathrm{i}\omega T_0)),\exp(\mathrm{i}\omega T_0)).\end{aligned} \tag{2.14}$$

The procedure that we propose below for the solution of the system (2.11) is accomplished by N successive steps. In the course of the nth step the function z_n and the T_n dependence of z_0 are determined, so that after N steps all the functions z_j ($j = 1 \ldots N$) and the dependence of z_0 on $T_1 \ldots T_N$ are determined. In view of this, the best way to present the procedure is by the method of mathematical induction.

Let us assume that a process of $n-1$ steps has been accomplished, in the course of which the functions z_j and the derivatives $\partial z_0/\partial T_j$ have been determined for $j = 1 \ldots n-1$, in such a manner that the functions z_j are a generalised power series in $\exp(\mathrm{i}\omega T_0)$. We can therefore calculate G_n, and the nth equation of the system (2.11) reads:

$$\partial z_0/\partial T_n + P_{n-1} + \partial z_n/\partial T_0 = G_n \tag{2.15}$$

where we introduced the notation

$$P_{n-1} = \sum_{j=1}^{n-1} \frac{\partial z_j}{\partial T_{n-j}} = \sum_{j=1}^{n-1} \left(\frac{\partial z_j}{\partial z_0} \frac{\partial z_0}{\partial T_{n-j}} + \frac{\partial z_j}{\partial \bar{z}_0} \frac{\partial \bar{z}_0}{\partial T_{n-j}} \right) \qquad P_0 = 0. \tag{2.16}$$

From the induction assumption it follows that P_{n-1} is already determined and is a generalised power series in $\exp(\mathrm{i}\omega T_0)$. As we demonstrated, z_0 is independent of T_0, so that equation (2.15) can be split into the following two equations:

$$\partial z_0/\partial T_n = \langle G_n - P_{n-1} \rangle \tag{2.17}$$

$$\partial z_n/\partial T_0 = G_n - P_{n-1} - \partial z_0/\partial T_n. \tag{2.18}$$

Since $\partial z_n/\partial T_0$ is constructed in such a manner that its T_0 average vanishes, z_n is a generalised power series in $\exp(\mathrm{i}\omega T_0)$. The solution of equation (2.18) is:

$$z_n = \int \left[G_n - P_{n-1} - \frac{\partial z_0}{\partial T_n} \right] \mathrm{d}T_0 + y_n(z_0, \bar{z}_0) \tag{2.19}$$

where y_n is an *arbitrary* function of z_0, $\bar{z}_0$. (For the sake of precision we shall choose the constant of integration so that $\langle z_n - y_n \rangle = 0$. Once y_n is chosen, z_n is determined by equation (2.19). The T_n dependence of z_0 is determined by equation (2.17), and this completes the nth step. Formally our procedure can be summarised as follows. The procedure is initialised by the computation of F_n ($n = 1 \ldots N$). Then the following steps are repeated for $n = 1 \ldots N$:

(i) Calculate $G_n(z_0, \bar{z}_0, \exp(\mathrm{i}\omega T_0))$ (equation (2.14)).
(ii) Calculate P_{n-1} (equation (2.16)).
(iii) Calculate $\partial z_0/\partial T_n$ (equation (2.17)).
(iv) Calculate z_n (equation (2.18)).

Collecting the equations of the steps (iii) we get a system of equations for $\partial z_0/\partial T_j$ ($j = 1 \ldots N$) which determines the $T_1 \ldots T_N$ dependence of z_0. We emphasise the fact that the N steps can be carried out without actually solving this system; its solution is a completely separate task. Though the equations (2.17) are in general nonlinear, in many cases, at least at low orders, they can be solved analytically. Even in cases for which no analytical solution is available, equations (2.17) have an advantage over the original equation (2.5) in that T_0 does not appear in them. Thus, the fast oscillations are eliminated from equations (2.17), a fact which greatly facilitates their numerical integration.

Equations (2.17) are solved successively, starting with $n = 1$. When the solution is completed, two constants of integration are left in z_0 (we remind the reader that z_0 is a complex quantity). These constants are fixed by the initial conditions.

As we have seen, in order to accomplish the expansion one has to choose the N functions y_j $(j = 1 \ldots N)$. Different choices lead, in general, to different expansions. This situation is analogous to the one encountered in former works in multivariable expansions, in which every expansion coefficient is determined up to an additive term proportional to the homogeneous solution of equation (1.2). These terms were discarded as a rule (Nayfeh 1973, Melvin 1977). One might guess that the analogous procedure in the present method would be to choose $y_j = 0$ (the null choice); however, this is not the case. It turns out that in order to get expansion coefficients of x free from the homogeneous terms one has to choose

$$y_i = (2\mathrm{i}\omega)^{-1}\, \partial z_0/\partial T_j. \tag{2.20}$$

In § 4, an expansion generated by the null choice is compared together with one generated by the choice (2.20) to an exact numerical calculation. The two expansions differ significantly, and the one generated by (2.20) is found to be superior.

It is evident that the choice of the functions y_j affects both the expansion coefficients z_j, $j = 1 \ldots N$ and the time dependence of z_0. However, it is worth noticing that the $n = 1$ equation of (2.17) remains invariant. Therefore only the T_1 dependence of z_0 can have certain physical interpretation. The dependence of z_0 on the higher time scales reflects mainly the particular choice of the functions y_j.

The quantities which appeared in this section have some general properties, which are discussed briefly in Appendix 2.

3. Example: the Duffing oscillator

As an illustration we apply the procedure developed in the previous section to the Duffing oscillator

$$\mathrm{d}^2x/\mathrm{d}t^2 + \omega^2 x = \epsilon x^3 \tag{3.1}$$

using the choice (2.20). The equation (3.1) takes the following form in terms of the complex variable z:

$$\mathrm{d}z/\mathrm{d}t = [\epsilon/(2\mathrm{i}\omega)^3)\exp(-\mathrm{i}\omega t)(\exp(\mathrm{i}\omega t)z - \exp(-\mathrm{i}\omega t)\bar{z})^3. \tag{3.2}$$

We shall carry on the expansion to second order, which means that we shall use the time scales T_0, T_1, T_2 and the expansion

$$z = z_0 + \epsilon z_1 + \epsilon^2 z_2. \tag{3.3}$$

Substituting equation (3.3) into the RHS of equation (3.2) and collecting the coefficients of ϵ and ϵ^2 we get F_1 and F_2 respectively:

$$F_1 = (\mathrm{i}/8\omega^3)[z_0^3 q^2 - \bar{z}_0^3 \bar{q}^4 + 3z_0\bar{z}_0^2\bar{q}^2 - 3z_0^2\bar{z}_0] \tag{3.4}$$

$$F_2 = (3\mathrm{i}/8\omega^3)[z_0^2 z_1 q^2 - \bar{z}_0^3\bar{z}_1\bar{q}^4 + 2z_0\bar{z}_0\bar{z}_1\bar{q}^2 + \bar{z}_0^2 z_1\bar{q}^2 - z_0^2\bar{z}_1 - 2z_0\bar{z}_0 z_1] \tag{3.5}$$

where $q \equiv \exp(\mathrm{i}\omega T_0)$.

We are now ready to execute the prescription given in § 2, starting with $n=1$:

(i) $G_1 = F_1$.

(ii) $P_0 = 0$.

(iii) $\partial z_0/\partial T_1 = \langle G_1 - P_0\rangle = -(3\mathrm{i}/8\omega^3) z_0^2 \bar{z}_0$ (3.6)

(iv) $z_1 = \int [G_1 - \partial z_0/\partial T_1]\,\mathrm{d}T_0 + (1/2\mathrm{i}\omega)\,\partial z_0/\partial T_1$

$= (1/32\omega^4)[2z_0^3 q^2 + \bar{z}_0^3 \bar{q}^4 - 6z_0\bar{z}_0^2\bar{q}^2 - 6z_0^2\bar{z}_0]$. (3.7)

We now repeat steps (i)–(iv) for $n=2$:

(i) $G_2 = (3\mathrm{i}/256\omega^7)(z_0^5 q^4 - 2z_0^4\bar{z}_0 q^2 + z_0^3\bar{z}_0^2 - z_0^2\bar{z}_0^3\bar{q}^2 + 2z_0\bar{z}_0^4\bar{q}^4 - \bar{z}_0^5\bar{q}^6)$ (3.8)

(ii) $P_1 = \dfrac{\partial z_1}{\partial z_0}\dfrac{\partial z_0}{\partial T_1} + \dfrac{\partial z_1}{\partial \bar{z}_0}\dfrac{\partial \bar{z}_0}{\partial T_1}$

$= (9\mathrm{i}/256\omega^7)(-2z_0^4\bar{z}_0 q^2 + 2z_0^3\bar{z}_0^2 - 2z_0^2\bar{z}_0^3\bar{q}^2 + z_0\bar{z}_6^4\bar{q}^4)$ (3.9)

(iii) $\partial z_0/\partial T_2 = \langle G_2 - P_1\rangle = (-15\mathrm{i}/256\omega^7) z_0^3\bar{z}_0^2$. (3.10)

(iv) $z_2 = \int (G_2 - P_1 - \partial z_0/\partial T_2)\,\mathrm{d}T_0 + (1/2\mathrm{i}\omega)\,\partial z_0/\partial T_2$

$= (1/1024\omega^8)(3z_0^5 q^4 + 24 z_0^4\bar{z}_0 q^2 - 30 z_0^3\bar{z}_0^2 - 30 z_0^2\bar{z}_0^3\bar{q}^2$

$+ 3z_0\bar{z}_0^4\bar{q}^4 + 2\bar{z}_0^5\bar{q}^6)$. (3.11)

The solution of equations (3.6) and (3.10) is

$$z_0 = \zeta \exp[\mathrm{i}\omega(1 - \tfrac{3}{8}s - \tfrac{15}{256}s^2)t + \mathrm{i}\phi] \tag{3.12}$$

where ζ and ϕ are integration constants to be determined by the initial conditions, and

$$s = \epsilon\zeta^2/\omega^4. \tag{3.13}$$

This completes the second-order MTSE for the Duffing oscillator. Our result agrees with previously derived expansions (Nayfeh 1973). The expansion can be carried further to higher orders, but the expressions for the expansion coefficients become increasingly complex. We found that if the expansion is carried on, the T_3 dependence of z_0 is given by

$$\partial z_0/\partial T_3 = -(123\mathrm{i}/8192\omega^{11}) z_0^4\bar{z}_0^3. \tag{3.14}$$

We see that even such simple expansions as that given above are rather laborious tasks for a hand calculation; however, they can be derived easily with a computer.

4. The effect of different choices of $y_j(z_0, \bar{z}_0)$ on the expansion

We do not intend to present here a comprehensive discussion of the subject. We shall rather restrict ourselves to a demonstration in the particular case of the Duffing oscillator. An MTSE of this oscillator with the choice (2.20) was presented in § 3. Using the null choice ($y_j = 0$) the computer generated the following expansion:

$$\exp(\mathrm{i}\omega t) z_0 = \zeta_1\sigma_1 \tag{4.1}$$

$$\epsilon \exp(\mathrm{i}\omega t) z_1 = (\zeta_1 s_1/32)(2\sigma_1^3 + \sigma_1^{-3} - 6\sigma_1^{-1}) \tag{4.2}$$

$$\epsilon^2 \exp(i\omega t) z_2 = (\zeta_1 s_1^2/1024)(3\sigma_1^5 + 60\sigma_1^3 + 2\sigma_1^{-5} + 21\sigma_1^{-3} - 138\sigma_1^{-1}) \tag{4.3}$$

where

$$\sigma_1 = \exp[i\omega(1 - \tfrac{3}{8}s_1 - \tfrac{51}{256}s_1^2)t + i\phi_1] \tag{4.4}$$

$$s_1 = \epsilon\zeta_1^2/\omega^4 \tag{4.5}$$

and ζ_1 and ϕ_1 are the integration constants to be fixed by the initial conditions. The MTSE for z to second order generated by the null choice is therefore

$$\begin{aligned}\exp(i\omega t) z^{(1)} &= \exp(i\omega t)(z_0 + \epsilon z_1 + \epsilon^2 z_2)\\ &= \zeta_1[\sigma_1 - (\tfrac{6}{32}s_1 + \tfrac{138}{1024}s_1^2)\sigma_1^{-1} + (\tfrac{1}{16}s_1 + \tfrac{15}{256}s_1^2)\sigma_1^3\\ &\quad + (\tfrac{1}{32}s_1 + \tfrac{21}{1024}s_1^2)\sigma_1^{-3} + \tfrac{3}{1024}s_1^2\sigma_1^5 + \tfrac{1}{512}s_1^2\sigma_1^{-5}].\end{aligned} \tag{4.6}$$

Using the formulae presented in § 3 we obtain the corresponding MTSE generated by the choice (2.20):

$$\begin{aligned}\exp(i\omega t) z^{(2)} &= \zeta_2[(1 - \tfrac{3}{16}s_2 - \tfrac{15}{512}s_2^2)\sigma_2 - (\tfrac{3}{16}s_2 + \tfrac{15}{512}s_2^2)\sigma_2^{-1}\\ &\quad + (\tfrac{1}{16}s_2 + \tfrac{3}{128}s_2^2)\sigma_3^2 + (\tfrac{1}{32}s_2 + \tfrac{3}{1024}s_2^2)\sigma_2^{-3} + \tfrac{3}{1024}s_2^2\sigma_2^5 + \tfrac{1}{512}s_2^2\sigma_2^{-5}]\end{aligned} \tag{4.7}$$

where

$$\sigma_2 = \exp[i\omega(1 - \tfrac{3}{8}s_2 - \tfrac{15}{256}s_2^2)t + i\phi_2] \tag{4.8}$$

$$s_2 = \epsilon\zeta_2^2/\omega^4. \tag{4.9}$$

Equating the coefficient of σ_1 in equation (4.6) to the coefficient of σ_2 in equation (4.7) we get

$$\zeta_1 = \zeta_2(1 - \tfrac{3}{16}s_s - \tfrac{15}{512}s_2^2) \tag{4.10}$$

and therefore

$$s_1 = s_2(1 - \tfrac{3}{8}s_2) + O(s_2^3). \tag{4.11}$$

Substituting this in equation (4.4) we get

$$\sigma_1 = \exp[i\omega(1 - \tfrac{3}{8}s_2 - \tfrac{15}{256}s_2^2 + O(s_2^3))t + i\phi_2] \tag{4.12}$$

in agreement with equation (4.8).

It can also be verified that the corresponding coefficients of the various powers of the σ in equations (4.7) and (4.6) differ by quantities of the order of s_2^3. We may therefore speculate that whenever both expansions converge they converge to the same limit, which is the exact solution. Nevertheless, the rates of convergence are considerably different. One may reach this conclusion simply by an inspection of the expressions for the σ in both cases: the coefficient of s_1^2 in equation (4.4) is roughly three times bigger than the corresponding coefficient of s_2^2 in equation (4.8). This tendency is even more pronounced in the next order: The coefficient of s_1^3 in the expansion of σ_1 is found to be $\frac{1419}{8192}$, compared with $\frac{123}{8192}$ in the expansion of σ_2 (equation (3.14)). In order to get a quantitative picture of this state of affairs, we have plotted in figure 1 both second-order expansions together with an exact numerical solution of the equation

$$d^2x/dt^2 + x = 0{\cdot}3x^3 \tag{4.13}$$

for some arbitrary initial conditions. The curves representing the expansion described by equations (4.7)–(4.9) and the numeric solution could not be resolved on the scale in

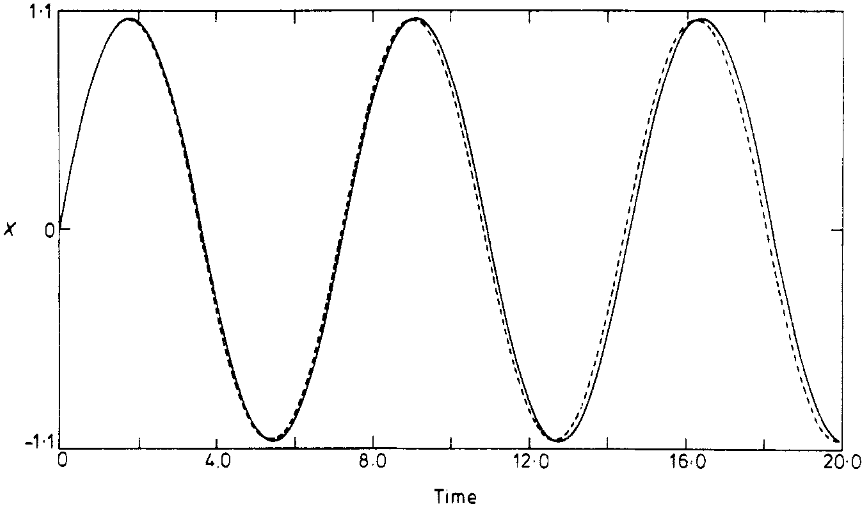

Figure 1. A comparison between an exact numerical integration and two different MTSEs of equation (5.13). ——— the numerical calculation and the MTSE generated by the choice (2.11); ------ the MTSE generated by the null choice.

which the figure is drawn; evidently, there was no difficulty in resolving these two curves from the third one.

The facts demonstrated in this section show that there is great practical importance in an intelligent use of the extra freedom, namely the choice of the functions y_j. This subject was not covered by the present study.

5. Discussion

The symbolic computer program for the generation of MTSEs has been written in the FORMAC language, and it carries out the N steps of the expansion procedure described in § 2. The program determines symbolically the functions $z_1 \ldots z_N$ and yields the derivatives $\partial z_0/\partial T_1 \ldots \partial z_0/\partial T_N$. The existing version does not solve equations (2.17) since it turns out that their analytic solution, if at all possible, is either trivial or considerably complicated. A computer-generated expansion for the van der Pol oscillator is presented in Appendix 1.

The computer memory volume required for the symbolic computation increases considerably with the order N of the expansion and the complexity of g. Therefore, although in principle the symbolic computation can be carried on to any order, an actual application is limited by the available size of the computer memory. The symbolic computations are also very time consuming and therefore very expensive. From these considerations the former procedures, whenever applicable, are much superior to ours. However, the symbolic program has, among others, one important advantage: it can handle functions $g(x, \mathrm{d}x/\mathrm{d}t)$ which contain one or more parameters, for instance $g = ax^2 + bx\ \mathrm{d}x/\mathrm{d}t$. The parameters a and b will appear explicitly in the computer-generated expansion coefficients. In addition, the FORMAC compiler supports rational arithmetic, and therefore the results are free from rounding errors.

Although one-dimensional oscillators like the Duffing and the van der Pol oscillators have numerous physical applications (for instance, Borenstein and Lamb 1972, Nayfeh 1968), there is naturally much greater interest in systems of coupled oscillators which may describe the vibration of molecules or mechanical constructions. There is also interest in such systems with explicit time dependence to describe forced vibrations (Sridhar *et al* 1975). In all these systems the occurrence of resonances has to be considered explicitly. This phenomenon is absent in oscillators of the type described by equation (1.2), and therefore it was not taken into account in the present treatment. A straightforward application of the present method to a system with resonances will produce terms with 'small denominators' and subsequently a nonconvergent perturbation series. Therefore, a generalisation of the present treatment to a system with resonances will necessarily have to contain new elements. Work in this direction is in progress.

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Appendix 1. Third-order computer-generated MTSE of the van der Pol oscillator with the choice (2.20)

The equation of motion of the van der Pol oscillator is (Nayfeh 1973)

$$d^2x/dt^2+\omega^2x=\epsilon(1-x^2)\,dx/dt. \qquad (A1.1)$$

By inspection of the computer-generated MTSE for this oscillator it was found that $\exp(i\omega t)z_n$ can be represented for $n=1, 2, 3$ as

$$\exp(i\omega t)z_n=\sum_{m=0}^{n-1}\omega^{2m-3n}\sum_{k=m-n}^{n-m+1}a_{mkn}z_0^{k+n-m}\bar{z}_0^{-k+n-m+1}\exp[i\omega t(2k-1)]. \qquad (A1.2)$$

The values of the coefficients a_{0kn}, a_{1kn} and a_{2kn} are given in tables 1, 2 and 3 respectively.

The function z_0 is determined by the initial conditions, and the equations

$$\partial z_0/\partial T_1=-(1/8\omega^2)z_0^2\bar{z}_0+\tfrac{1}{2}z_0 \qquad (A1.3)$$

$$\partial z_0/\partial T_2=(-7i/256\omega^5)z_0^3\bar{z}_0^2+((i/8\omega^3)z_0^2\bar{z}_0-(i/8\omega)z_0 \qquad (A1.4)$$

$$\partial z_0/\partial T_3=-(37/8192\omega^8)z_0^4\bar{z}_0^3+(35/1024\omega^5)z_0^3\bar{z}_0^2-(1/16\omega^4)z_0^2\bar{z}_0. \qquad (A1.5)$$

The solution of equation (A1.3) is discussed by several authors (for instance Nayfeh 1973).

Appendix 2. Some general features of the expansion

The various quantities which appear in § 2 have several well-defined and general features, which are worth a short discussion. In order to present them, we define the concept of the multivariable polynomial (MP) in the arguments $u_1 \ldots u_N$. A function

Table 1.

n \ k	−3	−2	−1	0	1	2	3	4
1			i/32	−i/16	i/16	−i/16		
2		−5/1536	−7/1024	−7/512	−7/512	1/128	−5/1024	
3	7i/24576	5i/8192	−23i/32768	37i/16384	−37i/16384	−3i/16384	−65i/98304	7i/18432

Table 2.

n \ k	−2	−1	0	1	2	3
1			i/4	−i/4		
2		−1/64	1/16	1/16	−1/128	
3	−5i/36864	i/2048	35i/2048	−35i/2048	19i/4096	−5i/6144

Table 3.

n \ k	−1	0	1	2
2		−1/16	−1/16	
3	i/512	−i/32	i/32	−i/256

$f(u_1 \ldots u_N)$ is an MP in $u_1 \ldots u_n$ ($f = \mathrm{MP}(u_1 \ldots u_N)$) if it can be presented as

$$f = \sum_{r_1, r_2, \ldots r_N} a_{r_1, r_2 \ldots r_N} u_1^{r_1} u_2^{r_2} \ldots u_N^{r_N} \tag{A2.1}$$

where $r_1 \ldots r_N$ are non-negative integers and $N \geqslant 1$. It is easy to establish that

$$h = \mathrm{MP}(\exp(\mathrm{i}\omega T_0) z_0, \exp(-\mathrm{i}\omega T_0) \bar{z}_0 \ldots \exp(\mathrm{i}\omega T_0) z_N, \exp(-\mathrm{i}\omega T_0) \bar{z}_N) \tag{A2.2}$$

$$F = \exp(-\mathrm{i}\omega T_0) \mathrm{MP}(\exp(\mathrm{i}\omega T_0) z_0, \exp(-\mathrm{i}\omega T_0) \bar{z}_0 \ldots \exp(\mathrm{i}\omega T_0) z_N, \exp(-\mathrm{i}\omega T_0) \bar{z}_N). \tag{A2.3}$$

If we choose the functions y_i to be of the form

$$y_j = z_i \mathrm{MP}(|z_0|)^2 \tag{A2.4}$$

then the following relations can be proved:

$$z_j = \exp(-\mathrm{i}\omega T_0) \mathrm{MP}(\exp(\mathrm{i}\omega T_0) z_0, \exp(-\mathrm{i}\omega T_0) \bar{z}_0) \tag{A2.5}$$

$$\partial z_0 / \partial T_j = z_0 \mathrm{MP}(|z_0|)^2 \tag{A2.6}$$

(equation (A2.6) is always valid for $j = 1$, regardless of equation (A2.4); therefore, the expansions generated by the choice (2.20) do have the properties of equations (A2.5), (A2.6)).

The form of equation (A2.6) suggests that a polar representation of z_0 would be useful. In the expansions of the conservative oscillators generated either by the null choice or by equation (2.20) it was found that the multivariable polynomial in equation (A2.6) has purely imaginary coefficients, and therefore z_0 is of the form

$$z_0 = \zeta \exp\left(\mathrm{i} \sum_{k=1}^{N} \alpha_k T_k + \mathrm{i}\phi\right) \tag{A2.7}$$

where α_k, ζ and ϕ are real.

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